Total variation as an adaptive signal dictionary

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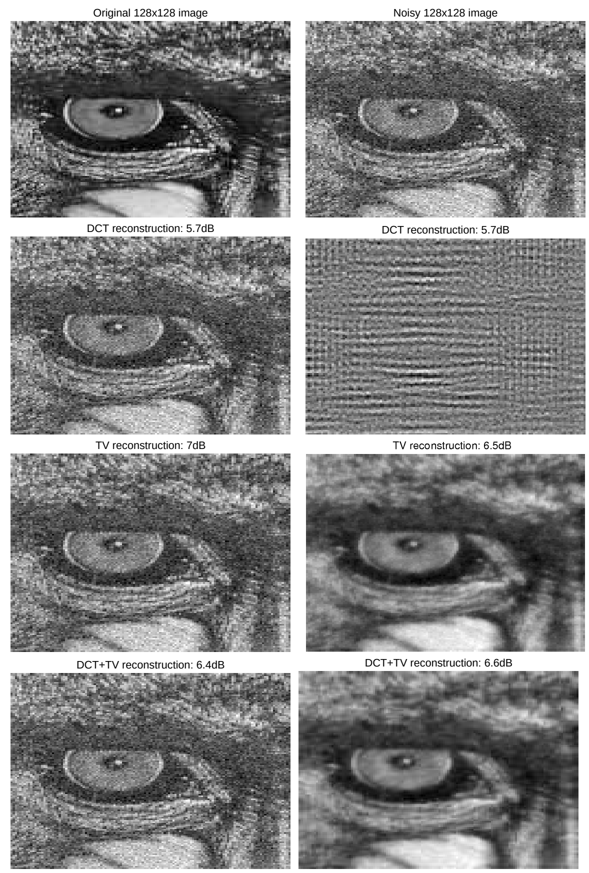
Representing a signal as a linear combination of a set of vectors is an important task in signal processing. Such a set of vectors form an orthogonal basis commonly called a dictionary. A dictionary's performance to represent a signal is data dependent. It's been shown in the 1-D case [1] that Total Variation (TV) can be used as a dictionary with excellent results for blocky signals. The 2-D case is non-trivial. A way to use TV as a dictionary in 2-D was found recently at LANL while developing the novel Iterative Reweighted Norm (IRN) algorithm [2]. We aim to evaluate the use of TV as a signal dictionary by itself, or in combination with other dictionaries, to solve the fundamental problem of image denoising.

Basis Pursuit (BP) and Total Variation (TV) regularization are techniques that, although with different philosophies, can be used for signal denoising. BP is a method for computing a sparse signal decomposition. BP denoising (BPDN) is done by allowing a misfit equal to the noise variance (between the original and reconstructed signals) when calculating the signal's sparse representation. BPDN involves the minimization over the vector of coefficients $\mathbf{u}$ of the function:

$$BPDN(\mathbf{u}) = \frac{1}{2}\|\Phi\mathbf{u}-\mathbf{b}\|_2^2 + \lambda\|\mathbf{u}\|_1\,, \qquad (1)$$

where $\mathbf{b}$ is the signal to be decomposed, Φ is the dictionary matrix, λ is a weighting factor controlling the relative importance of the data fidelity and sparsity terms (ℓ^2 and ℓ^1 terms respectively), and $\mathbf{u}$ is the sparse representation (coefficients). The resulting denoised signal $\mathbf{x}$ is then synthesized by using $\Phi\mathbf{u} = \mathbf{x}$.

On the other hand, TV denoising in 2-D is done by using a total variation penalized least squares



Example of denoising results using a combination of DCT and TV dictionaries. The top row shows the original (left) and noisy (right) images. The noisy image was denoised using two values of the parameter ratio $\lambda_{tv}/\lambda_{dct}$, one small (left column), one large (right column); this controls how much of the image's energy is processed by each dictionary. The second row shows the portion of the reconstructed image corresponding to the DCT dictionary. The third row shows the portion corresponding to the TV dictionary. The bottom row shows the complete reconstructed image.

functional:

$$T(\mathbf{x}) = \frac{1}{p}\|\mathbf{x}-\mathbf{b}\|_p^p + \frac{\lambda}{q}\left\|\sqrt{(D_x\mathbf{x})^2+(D_y\mathbf{x})^2}\right\|_q^q \qquad (2)$$

Note that in this functional, $\mathbf{x}$ denotes the objective denoised signal and **b** the input noisy signal. Again, λ is a weighting factor controlling the relative importance of the data fidelity and regularization terms. D_x and D_y are the discrete, horizontal and vertical derivative operators respectively. For the problem at hand, we consider the ℓ^2 TV functional ($p = 2, q = 1$). The resulting optimization problem is the minimization of the TV functional over the signal vector **x**.

In [1] an equivalence between these two techniques in 1-D was shown, allowing the characterization of a TV dictionary. The 2-D case is more complex. An equivalence is possible in 2-D if a linear, invertible operator could be found to represent the regularization term in the TV functional. Let us assume that such operator exists and that $\frac{\lambda}{q}\|\sqrt{(D_x\mathbf{x})^2+(D_y\mathbf{x})^2}\|_q^q = \mathbf{\Psi x}$. By a simple change of variables, $\mathbf{\Psi x} = \mathbf{u}$, the functionals in equations 1 and 2 become equivalent, making the pseudoinverse $\mathbf{\Psi}^+$ equivalent to a TV dictionary in 2-D.

The operator $\mathbf{\Psi}$ was derived while approximating the regularization term in the TV functional by a ℓ^2 term. This idea, borrowed from Iterative Re-weighted Least Squares (IRLS) is at the core of the novel IRN algorithm developed to solve a generic TV functional [2] and an algorithm called AST, to solve the BPDN functional [3]. This approximation allows us to solve the minimization of the TV functional by simply taking its derivative and equaling it to zero. The characterization of the operator $\mathbf{\Psi}$ and the approximation of the ℓ^1 term by a ℓ^2 term is accomplish by defining a matrix of weights, that changes over the algorithm iterations (thus the term *adaptive* to qualify the TV dictionary $\mathbf{\Psi}^+$). For details please refer to [2].

This newly established equivalence between the BPDN and the TV functionals in 2-D allowed us to use TV as a dictionary for image denoising, not only by itself but also in combination with other dictionaries. When using multiple dictionaries, a parameter λ is associated to each dictionary. The ratio between the different λ's controls the energy distribution between the dictionaries.

In theory, incorporating TV as a dictionary could potentially improve denoising of certain types of signals. This statement is a result of the known fact [1] that the suitability of a dictionary in a denoising problem is data-dependent. In a 2-D denoising problem, TV would contribute its ability to preserve edges on images with significant details.

Results

Denoising tests were run over different images using TV by itself and TV in combination with a Discrete Cosine Transform (DCT). For the case of TV as a dictionary alone, good results were obtained, particularly for blocky signals. However, for the case of TV in combination with a DCT dictionary the results were of lower quality than TV by itself (Fig. 1). Through different $\lambda_{tv}/\lambda_{dct}$ ratios, changes on the image energy distribution between the two dictionaries are observed, but the combined result is always of lower quality than TV by itself. Note that factors such as the image size and the choice of λ might influence the results obtained.

Future work

An optimized version of the code will be developed, allowing us to test bigger images and different λ values. Also, we will incorporate new dictionaries to use in combination with TV.

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